

# Remark on Shape Invariant Potential

Elso Drigo Filho<sup>1</sup>

Instituto de Biociências, Letras e Ciências Exatas-UNESP  
Departamento de Física  
Rua Cristovão Colombo, 2265  
15055 São José do Rio Preto, SP, Brazil

Regina Maria Ricotta

Faculdade de Tecnologia de São Paulo, CEETPS-UNESP  
Praça Fernando Prestes, 30  
01121-060 São Paulo, SP, Brazil

## ABSTRACT

The usual concept of shape invariance is discussed and one extension of this concept is suggested.

Gedenshtein [1] defined the “shape invariant” potentials by the relationship

$$V_+(x; a_0) - V_-(x; a_1) = W^2(x, a_0) + W'(x, a_0) - W^2(x; a_1) + W'(x; a_1) = R(a_1) \quad (1)$$

where  $W(x; a)$  is the superpotential,  $a_0$  and  $a_1$  stand for parameters of the supersymmetric partner potentials  $V_+$  and  $V_-$ ,  $R(a)$  is a constant. The supersymmetric partners are related with the supersymmetric Hamiltonian in an usual way [2],  $V_+ = W^2 - W'$  and  $V_- = W^2 + W'$ .

The relationship between shape invariance and solvable potentials is discussed by several authors (see, for instance, [3] and [4]). Other mathematical aspects of shape invariant potentials are also present in the literature, for example in the supersymmetric WKB approximation, [5], Berry phase, [6], and in the path-integral formulation, [7].

There is a general conclusion about these kind of potentials which is that the concept of shape invariance is a sufficient but not a necessary condition for the potential to become exactly solvable, [4].

In a recent work, [8], the Hulthén potential was studied from the Supersymmetric Quantum Mechanics formalism. This potential has an interesting property, that is when the angular momentum is zero,  $l = 0$ , it is not shape invariant in the sense expressed in ref.[1]. However, it is still possible to construct a general form of the potentials in the super-family of Hamiltonians:

$$V_n(r) - E_0^{(n)} = W_n^2(r) - \frac{d}{dr}W_n(r) = \frac{n(n-1)\delta^2 e^{-2\delta r}}{2(1-e^{-\delta r})^2} - \frac{[n(1-n)\delta + 2]\delta e^{-\delta r}}{2(1-e^{-\delta r})} + \frac{1}{2}\left(-\frac{n}{2}\delta + \frac{1}{n}\right)^2. \quad (2)$$

---

<sup>1</sup>Work partially supported by CNPq and FAPESP

where  $n = 1, 2, 3, \dots$  labels the  $n$ -th member of the super-family whose ground-state is  $E_0^{(n)}$ , ( $n = 1$  and  $2$  correspond to the two first members  $V_+$  and  $V_-$ , respectively, except by additive constants in  $V_-$ ) and  $\delta$  is a fixed parameter. For  $n = 1$  the potential in (2) leads us to the usual Hulthén potential  $V_H$

$$V_+(r) = V_H(r) - E_0^{(1)} = -\frac{\delta e^{-\delta r}}{1 - e^{-\delta r}} + \frac{1}{2}\left(\frac{1 - \delta}{2}\right)^2, \quad (3)$$

and from (2) it is easy to note that the condition (1) is not satisfied, i.e.,  $V_H$  is not shape invariant but the whole super-family has the same functional form given by equation (2).

Taking the previous example, it is possible to suggest an extension of the concept of shape invariance. This invariance would be associated with the functional form of the whole super-family potentials and not only with the first two members ( $V_+$  and  $V_-$ ), since all the members of super family can be written in a general functional form in terms of one or more parameters (as the natural number  $n$  in Hulthén potential case). In other words, it is possible to construct a general expression for all potentials of the super-family.

The simple example of the free particle in a box can be used to make clear the above idea. The Hamiltonian  $H$  in this case is

$$H_+ = H - E_0^{(1)} = -\frac{d^2}{dx^2} - 1; \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \quad (4)$$

where the constant term  $(-1)$  sets the eigenvalue of the ground state of  $H_+$  to zero, [9]. In this case the general form for the superpotential is

$$W_n(x) = n \tan(x) \quad (5)$$

where  $n$  is a natural number different from zero, ( $n = 1, 2, 3, \dots$ ). The super-family is such that  $E_n^{(1)} = n^2$  and the  $n$ -th member of the super-family potential is

$$V_n(x) - E_0^{(n)} = \frac{n(n-1)}{\cos^2(x)} - n^2. \quad (6)$$

Thus, it is not shape invariant in the Gendenshtein's sense, [1], since  $V_+ = -1$  and  $V_- = \frac{2}{\cos^2(x)} - 1$ , whereas it is shape invariant in the extended sense.

In our definition the potentials are shape invariant when it is possible to construct a super-family whose members have the same functional form. On the other hand, in the usual definition introduced by Gendenshtein, once relation (1) is satisfied it is possible to find all the members of the super-family. However, having built a super-family it does not necessarily mean that relation (1) is satisfied, as shown in the two examples above of the Hulthén potential and the particle in a box. In other words, Gendenshtein's condition of shape invariance is sufficient but not a necessary condition to obtain the super-family.

The interesting question to be studied now is if the extended shape invariance is a necessary condition to the potential to be exactly solvable. Other questions concerning shape invariance, [5], [6], [7], can also be analysed using this extended concept.

We acknowledge Dr. G. Levai for a critical reading of the manuscript. One of us (E.D.F.) would like to thank Prof. A. Inomata for calling his attention to the non-shape invariance problem of the Hulthén potential.

## References

- [1] L. Gedenshtein, JETP Lett. **38** (1983) 356
- [2] F. Cooper and B. Freedman, Ann. Phys. NY **146** (1983) 262; R. W. Haymaker and A. R. Rau, Am. J. Phys. **54** (1986) 928
- [3] G. Levai, J. Phys. A: Math. Gen. **25** (1992) L521; G. Levai Lect. Notes in Phys. **427** Ed. H. V. van Gevamb Spring-Verlag (Berlin) (1993) 107; T.Fukuiaand and A. Aizawa, Phys. Lett. **A180** (1993) 308
- [4] F. Cooper, J. N. Ginocchio and A. Khare, Phys. Rev. **D36** (1987) 2458; Cao X. C., J. Phys. A: Math. Gen. **24** (1991) L1165
- [5] R. Dutt, A. Khare and U. P. Sukhatme, Phys. Lett. **B181** (1986) 295; D. T. Barclay and C. J. Maxwell, Phys. Lett. **A157** (1991)
- [6] D. Bhaumik, B. Dutta-Roy, B. K. Bagchi and A. Khare, Phys. Lett. **A193** (1994) 11
- [7] R. De, R. Dutt and U. Sukhatme, Phys. Rev. **A46** (1992) 6869
- [8] E. Drigo Filho and R. M. Ricotta, Mod. Phys. Lett. **A10** (1995) 1613
- [9] E. Drigo Filho, Rev. Bras. Fis. **20** (1990) 258